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Scalarization and Nonlinear Scalar Duality for Vector Optimization with Preferences that are not necessarily a Pre-order Relation

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Abstract. We consider problems of vector optimization with preferences that are not necessarily a pre-order relation. We introduce the class of functions which can serve for a scalarization of these problems and consider a scalar duality based on recently developed methods for non-linear penalization scalar problems with a single constraint.

Key words. duality, preferences, scalarization, vector optimization.

1. Introduction

Problems of vector (multi-criteria) optimization arise when there are some different criteria for the choice of a preferable object. As a rule it is assumed that the totality of these criteria forms a pre-order relation. The theory of vector optimization with respect to (w.r.t.) pre-order relation is well developed (see, for example, [5, 11]). However, often we get preferences that form a relation, which is not a pre-order. Let us give some simple examples. Assume that we have $m>1$ criteria (objective functions) $f_1,\dots,f_m$ defined on a set X. Each element $x\in X$ can be estimated by a vector of numbers $(f_1(x),\dots,f_m(x))$. Usually it is assumed that x is more preferable that $y(x\succeq y)$ if $f_i(x)\geqslant f_i(y)$ for all $i\in I=\{1,\dots,m\}$. Clearly $\succeq$ is a pre-order relation. However, sometimes we need different kind of preferences, which are either weaker or stronger than $\succeq$. For example, let $m>2$ and $I_1=\{2,\dots,m\}$, $I_m=\{1,\dots,m-1\}$. Consider preferences $\succeq_1$ defined in the following way: $x\succeq_1 y$ if either $f_i(x)\geqslant f_i(y)$ for $i\in I_1$ or $f_i(x)\geqslant f_i(y)$ for $i\in I_m$. The preferences $\succeq_1$ are weaker than $\succeq$ (i.e. $x\succeq y$ implies $x\succeq_1 y$) and these preferences are not transitive, so $\succeq_1$ is not a pre-order relation. Consider now another preferences $\succeq_2$. We say that $x\succeq_2 y$ if $f_i(x)\leqslant f_i(y)$ for all $i\in I$ and either $f_1(x)-f_1(y)\geqslant f_2(x)-f_2(y)$ or $f_2(x)-f_2(y)\geqslant f_3(x)-f_3(y)$. Clearly $\succeq_2$ is stronger than $\succeq$ and $\succeq_2$ is not a pre-order relation. Both relations $\succeq_1$ $(i=1,2)$ have the following structure: $x\succeq_i y$ means that the vector $(f_1(x)-f_1(y),\dots,f_m(x)-f_m(y))$ belongs to a conic set K_i. This set can be represented as the union of two convex cones, however K_i itself is not a convex cone.

Note that preferences that are not pre-order relations have been studied in mathematical economics (see for example, [6] and references therein).

In this paper we study the class of preferences that are defined by means the so-called strongly star-shaped conic sets in a Banach space X. This is a large class of preferences that can be successfully examined. The simplest example of a strongly star-shaped conic set is the union K of a finite number of convex closed cones $K_i(i \in I)$ such that the intersection $\bigcap_{i \in I} \mathrm{int} K_i$ is not empty. Each strongly star-shaped set K determines the relation $\geqslant_K$ on X, where $x \geqslant y \Longleftrightarrow x - y \in K$. If K is not convex the $\geqslant_K$ is not a pre-order relation.

The relation $\geqslant_K$ generates different types of minimality. We restrict ourselves by weakly minimal, minimal and properly minimal points. The examination of various types of minimal points is the subject of vector optimization. One of the most popular approaches in vector optimization is to use a scalarization of preferences. We suggest a certain class of functions that provide a scalarization of relations $\geqslant_K$. (Sublinear functions from the this class have been studied before (see, for example [5, 7]. They can be used for description of pre-order relations generated by convex cones.) We study properties of functions from this class and provide some examples. Using this class we construct scalar optimization problems such that weakly minimal points, minimal points and properly minimal points can be completely described as solutions of these problems. Duality for these scalar optimization problems can be considered as a certain scalar duality for the initial problem of vector optimization. We discuss this form of the duality and give sufficient conditions for the validness of the zero duality gap property.

Different approaches to vector optimization duality can be found in literature. In particular, some authors (see, for example [3, 4]) suggest to formulate a dual problem for a vector optimization problem as also a problem of vector optimization. In such a case the dual problem is in a certain sense symmetrical to a primal one. However, the proposed scalar duality is much simpler than vector one.

The paper has the following structure. In Section 2 we provide some brief preliminary definitions and results related to strongly star-shaped conic sets. Functions $p_{u,K}$ that serve for a scalarization of the relation $\geqslant_K$ are introduced and studied in Section 3. Some examples of these functions are given in Section 4. Weakly minimal, minimal and properly minimal points and their characterization by means of functions $p_{u,K}$ are examined in Section 5. Pre-order relations generated by convex cones are considered in Section 6. Scalarization of vector optimization problems presented in Section 7. Scalar duality for these problems is defined and examined in Section 7.

2. Preliminaries

Let X be a Banach space and $K \subset X$. We denote by $\mathrm{int} K$, $\mathrm{bd} K$, $\mathrm{cl} K$ the interior of K, the boundary of K and the closure of K, respectively. For each $x \in X$

denote by R_x the ray starting at zero and going through x: $R_x = \{\lambda x: \lambda \geqslant 0\}$. The following definition (see, for example [8]) plays a key role in the sequel. A set K is called strongly star-shaped if there exists a point $u \in \operatorname{int} K$ such that the ray $u + R_x$ does not intersect the boundary $\operatorname{bd} K$ of the set K more than once for each $x \in X$. The set all points u, which enjoy this property is denoted by $\operatorname{kern}_* K$ (see, for example, [8]). Thus, if $u \in \operatorname{kern}_* K$ then for each x, the ray $u + R_x$ either intersect the boundary of K once or does not intersect this boundary. The latter means that $u + R_x \subset \operatorname{int} K$.

A set $K \subset X$ is called star-shaped if there exists a point $u \in K$ such that $\alpha u + (1-\alpha)x \in K$ for all $x \in K$ and $\alpha \in (0,1)$. The set of all points u which posses this property is denoted by $\operatorname{kern} K$. A strongly star-shaped set is star-shaped and $\operatorname{kern}_* K \subset \operatorname{kern} K$. This fact is well-known. Its proof for finite-dimensional case can be found for example in [8], however this proof is valid for an arbitrary Banach space.

The set K is called radiative, if $0 \in \operatorname{kern}_* K$. The main tool for the examination of a radiative set K is its Minkowski gauge μ_K. By definition

$$\mu_K(x) = \inf\{\lambda > 0: x \in \lambda K\}.$$

The following result holds:

THEOREM 2.1. *If K is a radiative set then its Minkowski gauge μ_K is continuous.*

A proof can be found, for example in [8], Proposition 5.10. Only finite dimensional situation was considered in [8], however this proof holds for Banach spaces.

If K is a closed radiative set then $K = \{x: \mu_K(x) \leqslant 1\}$. Since μ_K is continuous and $\mu_K(0) = 0$ it follows that $0 \in \operatorname{int} K$.

Consider a strongly star-shaped set K. If $u \in \operatorname{kern}_* K$ then the set $K - u$ is radiative, hence $0 \in \operatorname{int}(K-u)$. As it follows from the definition, $\operatorname{kern}_* K \subset \operatorname{int} K$.

Recall that a set $K \subset X$ is called conic if $x \in K \Longrightarrow (R_x \setminus \{0\}) \subset K$. Let K be a conic strongly star-shaped set. It is easy to check that the set $\operatorname{kern}_* K$ is a conic set. Denote by $U(K)$ the set of points $u \in K$, which possess the following properties:

(1) $u \in \operatorname{kern}_* K$;
(2) for each $x \in X$ the line $x + \{\lambda u: \lambda \in \mathbb{R}\}$ is not contained in K.

It is easy to check that the set $U(K)$ is conic. We now give some examples. If K is a convex one, $K \neq X$ and $\operatorname{int} K$ is nonempty, then $\operatorname{kern}_* K = U(K) = \operatorname{int} K$. If $K = X$, then $\operatorname{kern}_* K = X$, however $U(K) = \emptyset$. We shall show (see Corollary 4.1) that the set $U(K)$ is nonempty if K is a finite union of convex cones K_i such that the intersection $\bigcap_i \operatorname{int} K_i$ is nonempty.

Denote by $\mathcal{K}(X)$ the set of all conic closed sets $K \subset X$ with non empty $U(K)$. Let $K \in \mathcal{K}(X)$. A conic set K generates the relation $\geqslant_K$. By definition $x \geqslant_K y$

$\Longrightarrow x-y\in K$. If K is a convex cone then $\geqslant_K$ is a pre-order relation. We say that x is greater than y and write $x>_K y$ if $x-y\in K\setminus\{0\}$. We write $x\geqslant_K y$ if $x-y\in K$.

We now indicate some properties of the relation $\geqslant_K$ for $K\in\mathcal{K}(X)$.

PROPOSITION 2.1. *Let $u\in U(K)$. Then*

(1) *$\nu x+\lambda u\in K$ for each $x\in K$ and $\lambda>0$, $\nu>0$.*
(2) *For each $x\in X$ there exist $\lambda>0$ and $\nu\leqslant 0$ such that $x\leqslant_K \lambda u$ and $x\geqslant_K \nu u$.*
(3) *for each $x\in X$ the set $\Lambda_x=\{\lambda\in\mathbb{R}\colon \lambda u\geqslant_K x\}$ is a closed segment of the form $[\lambda_x,+\infty)$ with $\lambda_x>-\infty$.*

Proof. (1) We prove a stronger assertion: if K is a star-shaped conic set and $u\in\operatorname{kern}K$, then $\nu x+\lambda u\in K$ for each $x\in K$ and $\lambda>0$, $\nu>0$. First assume that $\nu=1$. Let $x\in K$ and $\lambda>0$. Let $\alpha=\frac{1}{1+\lambda}$. Since $u\in\operatorname{kern}K$ it follows that $\alpha(x)+(1-\alpha)u\in K$, hence $x+\frac{1-\alpha}{\alpha}u=x+\lambda u\in K$. Since K is a conic set, we can easily extend this result for an arbitrary $\nu>0$.

(2) Let $x\in X$. Since $u\in\operatorname{int}K$ it follows that there exists $\varepsilon>0$ such that $u-\varepsilon x\in K$ and $u+\varepsilon x\in K$. Let $\lambda=1/\varepsilon$, then $\lambda u-x\in K$ and $\lambda u+x\in K$ hence $x\leqslant_K\lambda u$ and $x\geqslant_K(-\lambda)u$.

(3) Let $x\in X$. Due to (2) the set Λ_x is nonempty. Let $\lambda\in\Lambda_x$ and $\mu>\lambda$. We have $\mu u-x=(\mu-\lambda)u+(\lambda u-x)$. Since $\lambda u-x\in K$ and $\mu-\lambda>0$, we can apply item (1) of this proposition, which shows that $\mu u-x\in K$. We have demonstrated that $(\lambda\in\Lambda_x,\mu>\lambda)\Longrightarrow\mu\in\Lambda_x$. It follows from this that Λ_x is unbounded from above segment. Since K is closed it follows that the segment Λ_x is closed. Due to the definition of $U(K)$, we conclude that the line $x+\{\lambda u\colon \lambda\in\mathbb{R}$ does not lie in K. It implies that the set Λ_x is bounded from below. $\square$

3. Functions $p_{u,K}$ and their Properties

Let $\bar{\mathbb{R}}$ be the extended real line: $\bar{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$. A function $p\colon X\to\bar{\mathbb{R}}$ is called positively homogeneous if $p(\lambda x)=\lambda p(x)$ for all $x\in X$ and $\lambda>0$.

For an arbitrary conic set K and $u\in K$ define the function $p_{u,K}\colon X\to\bar{\mathbb{R}}$ by setting

$$p_{u,K}(x)=\inf\{\lambda\in\mathbb{R}\colon \lambda u-x\in K\},\tag{3.1}$$

where the infimum over the empty set is equal to $+\infty$. It is easy to see that $p_{u,K}$ is positively homogeneous.

If $K\in\mathcal{K}(X)$ and $u\in U(K)$ then the function $p_{u,K}$ enjoys some good properties. In such a case $p_{u,K}(x)=\lambda_x$, where λ_x is the left end-point of the segment Λ_x from Proposition 2.1 (3), hence $p_{u,K}$ is finite, also the infimum in (3.1) is attained: $p_{u,K}(x)=\min\{\lambda\colon \lambda u-x\in K\}$. The latter implies the inclusion $p_{u,K}(x)u-x\in K$.

The following properties of the function $p_{u,K}$ defined by (3.1) will be useful in the sequel.

PROPOSITION 3.1. *Let $K \in \mathcal{K}$. Then*:

$$p_{u,K}(x+\mu u) = p_{u,K}(x) + \mu, \quad x \in X, \ \mu \in \mathbb{R}. \tag{3.2}$$

and

$$\{x \colon p_{u,K}(x) \leqslant \mu\} = \mu u - K, \quad \mu \in \mathbb{R}. \tag{3.3}$$

Proof.

(1) Let $x \in X$ and $\mu \in \mathbb{R}$. For $\lambda \in \mathbb{R}$ denote $\lambda_\mu = \lambda - \mu$. We have

$$\begin{aligned} p_{u,K}(x+\mu u) &= \min\{\lambda \in \mathbb{R} \colon \lambda u - (x - \mu u) \in K\} \\ &= \min\{\lambda_\mu + \mu \colon \lambda_\mu \in \mathbb{R}, \lambda_\mu u - x \in K\} \\ &= \mu + \min\{\lambda_\mu \in \mathbb{R} \colon \lambda_\mu u - x \in K\} = \mu + p_{u,K}(x). \end{aligned}$$

(2) Let $x \in \mu u - K$ and $k = \mu u - x$. Then $k \in K$. We have

$$\begin{aligned} p_{u,K}(x) &= \min\{\lambda \colon x \leqslant \lambda u\} = \min\{\lambda \colon \mu u - k \leqslant \lambda u\} \\ &= \min\{\lambda \colon -k \leqslant (\lambda - \mu) u\} \\ &= \mu + \min\{\lambda' \colon -k \leqslant \lambda' u\}. \end{aligned}$$

Since $-k \leqslant 0 \cdot u$ it follows that $p_{u,K}(x) \leqslant \mu$. Assume now that $p_{u,K}(x) \leqslant \mu$. Then $\min\{\lambda \colon x \leqslant_K \lambda u\} \leqslant \mu$. It follows from Proposition 2.1 (3) that $x \leqslant_K \mu u$. □

We also need the following simple assertion.

PROPOSITION 3.2. *Let $K \in \mathcal{K}(X)$ and $K \setminus \{0\} \subset \operatorname{int} L$, where L is a conic set. Let $u \in U(K)$. Then $p_{u,K}(x) < p_{u,L}(x)$ for all $x \neq cu$ with $c \in \mathbb{R}$.*

Proof. Let $x \neq cu$ with $c \in \mathbb{R}$. Consider the point $y = p_{u,K}(x)u - x$. Then $y \neq 0$. It follows from Proposition 2.1 (3) that $y \in K$. Since $K \setminus \{0\} \subset \operatorname{int} L$, we can find $\varepsilon > 0$ such that $(p_{u,K}(x) - \varepsilon)u - x \in L$. This means that $p_{u,L}(x) < p_{u,K}(x)$. □

For each $\gamma \in \mathbb{R}$ consider the upper level set $M_\gamma = \{x \colon p_{u,K}(x) \geqslant \gamma\}$ of the function $p_{u,K}$. Clearly $\bigcup_{\gamma \in \mathbb{R}} M_\gamma = X$ and $M_{\gamma 2} \subset M_{\gamma 1}$ if $\gamma_1 < \gamma_2$. Due to (3.2) we have

$$M_\gamma = \{x \colon p_{u,K}(x - \gamma u) \geqslant 0\}.$$

It follows from the definition of $\operatorname{kern}_* K$ and $U(K)$ that the set $K - u$ is radiative for all $u \in U(K)$, hence we can consider the Minkowski gauge of this set.

PROPOSITION 3.3. *Let $\gamma \in \mathbb{R}$. Then $p_{u,K}(x - \gamma u) = \mu_{K-u}(-(x - \gamma u))$ for all $x \in M\gamma$, where μ_{K-u} is the Minkowski gauge of the set $K - u$.*

Proof. Let $x \in M\gamma$. Since $p_{u,K}(x-\gamma u) \geqslant 0$ and

$$p_{u,K}(x-\gamma u) = \inf\{\lambda \in \mathbb{R}\colon\ \lambda u - (x-\gamma u) \in K\}$$

it follows that

$$p_{u,K}(x-\gamma u) = \inf\{\lambda > 0\colon\ \lambda u - (x-\gamma u) \in K\}$$

We have

$$\begin{aligned}\mu_{K-u}(\gamma u - x) &= \inf\{\lambda > 0\colon\ \gamma u - x \in \lambda(K-u)\} = \inf\left\{\lambda > 0\colon\ \frac{\gamma u - x}{\lambda} \in K - u\right\}\\ &= \inf\left\{\lambda > 0\colon\ \frac{\gamma u - x}{\lambda} + u \in K\right\} = \inf\{\lambda > 0\colon\ \lambda u - (x-\gamma u) \in K\}\\ &= p_{u,K}(x-\gamma u).\end{aligned}$$

COROLLARY 3.1. *The function $p_{u,K}$ is continuous.*

Proof. It follows from Proposition 3.3 and Theorem 2.1. □

We now show that each continuous positively homogeneous function, for which there exists an element $u \in X$ such that

$$p(x+\mu u) = p(x) + \mu, \quad x \in X,\ \mu \in \mathbb{R}. \tag{3.4}$$

can be presented in the form $p_{u,K}$ with some K.

THEOREM 3.1. *Let p be a continuous positively homogeneous function such that* (3.4) *holds for some $u \in X$. Let*

$$K = \{-x\colon\ p(x) \leqslant 0\}. \tag{3.5}$$

Then $u \in U(K)$ and $p = p_{u,K}$.

Proof. Let K be defined by (3.5). Since p is positively homogeneous it follows that K is a conic set. Since p is continuous it follows that the set K is closed. We have $-K = \{x\colon\ p(x) \leqslant 0\}$. The interior $\mathrm{int}(-K)$ of this set coincides with $\{x\colon\ p(x) < 0\}$. Indeed, the set $\{x\colon\ p(x) < 0\}$ is open, hence this set is contained in $\mathrm{int}(-K)$. Let $x \in \mathrm{int}(-K)$. Then $x + \varepsilon u \in -K$ for a small enough $\varepsilon > 0$. Hence $p(x+\varepsilon u) = p(x) + \varepsilon \leqslant 0$, so $p(x) < 0$. It follows from aforesaid that $\mathrm{bd}(-K) = \{x\colon\ p(-x) = 0\}$. Let us show that $-u \in \mathrm{kern}_*(-K)$. Indeed, let $x \in X$ and $y \in -u + R_x$. Then there exists $t \geqslant 0$ such that $y = -u + tx$. A point y belongs to the boundary of $-K$ if and only if

$$p(y) = p(-u+tx) = -1 + p(tx) = -1 + tp(x) = 0.$$

If $p(x) \neq 0$ then there is the unique t such this holds (the ray $-u + \mathbb{R}_x$ intersects the boundary once). The case $p(x) = 0$ leads to the equality $-1 = 0$, which means that case is impossible (the ray $-u+R_x$ does not intersect the boundary). Thus, the ray $-u+R_x$ does not intersect the boundary $\mathrm{bd}(-K)$ more than once: $-u\in \mathrm{kern}_*(-K)$. This implies $u\in \mathrm{kern}_* K$. We now show that the line $L_x=x+\{\lambda u\colon u\in \mathbb{R}\}$ is not located in K. Indeed, if $L_x\subset K$ then for each $\lambda\in\mathbb{R}$ we have $p(\lambda u+x)=\lambda p(x)\leqslant 0$, which is impossible. We have demonstrated that $u\in U(K)$.

Since $u\in U(K)$, we can define the function $p_{u,K}$. We now check that $p=p_{u,K}$. Indeed

$$\begin{aligned} p_{u,K}(x) &= \min\{\lambda\colon \lambda u-x\in K\}=\min\{\lambda\colon x-\lambda u\in -K\}\\ &= \min\{\lambda\colon p(x-\lambda u)\leqslant 0\}=\min\{\lambda\colon p(x)-\lambda\leqslant 0\}\\ &= \min\{\lambda\colon p(x)\leqslant\lambda\}=p(x). \end{aligned}$$ □

Remark 3.1. Condition (3.4) have been used in the literature for the definition of topical functions (see for example, [9] and references therein).

Let $u\in X$. Denote the set of all continuous positively homogeneous functions $p\colon X\to\mathbb{R}$ such that (3.4) holds with the given u by $\mathcal{P}_u$. We have checked that each function $p\in\mathcal{P}_u$ has the form $p=p_{u,K}$ with a conic set K such that $K\in\mathcal{K}(X)$ and $u\in U(K)$. We now describe some properties of the set $\mathcal{P}_u$.

PROPOSITION 3.4. *The set $\mathcal{P}_u$ possess the following properties*:

(1) *$\mathcal{P}_u$ is convex*;
(2) *$\mathcal{P}_u$ is a lattice with respect to the pointwise order relation, that is the functions $\bar{p}(x)=\max_{i=1,\dots,m}p_i(x)$ and $\underline{p}(x)=\min_{i=1,\dots,m}p_i(x)$ belong to $\mathcal{P}_u$, if $p_1,\dots,p_m$ belong to $\mathcal{P}_u$*;
(3) *$\mathcal{P}_u$ is closed w.r.t uniform convergence.*

We omit the simple proof of this proposition.

Let $K\in\mathcal{K}(X)$. Denote by CK the complement to the cone $\mathrm{int}K$: $CK=\{x\in X\colon x\notin \mathrm{int}K\}$. If $u\in U(K)$ then $\mathrm{int}K=\{x\in X; p_{u,K}(-x)<0\}$, hence $CK=\{x\in X\colon p_{u,K}(-x)\geqslant 0\}$. It is easy to check that $\mathrm{int}\,CK=\{x\in X\colon p_{u,K}(-x)>0\}$.

Let $u\in U(K)$ and $\Lambda_x=\{\lambda\in\mathbb{R}\colon \lambda u-x\in K\}$. Then due to Proposition 2.1 there exists a number λ_x such that $\Lambda_x=[\lambda_x,+\infty)$. We have $p_{u,K}(x)=\lambda_x$. This implies the following:

$$\begin{aligned} \sup\{\nu\colon \nu u-x\in CK\} &= \sup\{\nu\colon \nu u-x\notin K\}\\ &= \inf\{\lambda\colon \lambda u-x\in K\}=p_{u,K}(x). \end{aligned} \tag{3.6}$$

PROPOSITION 3.5. *Let $u\in U(K)$. Then $p_{-u,CK}=-p_{u,K}$.*

Proof. Applying (3.6) we conclude that

$$\begin{aligned} p_{u,K}(x) &= \sup\{\nu\colon \nu u-x\in CK\}=\sup\{-\lambda\colon -\lambda u-x\in CK\}\\ &= -\inf\{\lambda\colon -\lambda u-x\in CK\}=-\inf\{\lambda\colon \lambda(-u)-x\in CK\}\\ &= -p_{-u,CK}(x). \end{aligned}$$ □

Consider now the conic set $-K$.

PROPOSITION 3.6. *If* $K \in \mathcal{K}(X)$ *then* $-K \in \mathcal{K}(X)$ *and* $U(-K) = -U(K)$.

Proof. Since $u + \lambda x \in K \Longleftrightarrow -u + \lambda(-x) \in (-K)$ it follows that $\text{kern}_*(-K) = -\text{kern}_* K$. Clearly the line $x + \{\lambda u \colon \lambda \in \mathbb{R}\}$ is not contained in K if and only if the line $-x + \{\lambda(-u) \colon \lambda \in \mathbb{R}\}$ is not contained in $-K$. This the result follows. $\square$

COROLLARY 3.2. *We have* $U(-K) \subset U(CK)$. *Indeed, if* $-u \in U(-K)$ *then* $u \in U(K)$. *Applying Proposition* 3.5 *and Theorem* 3.1 *we conclude that* $u \in U(CK)$.

4. Examples

First we give an example of a conic set $K \in \mathcal{K}(X)$, which is non-convex. For this purpose we prove the following assertion.

PROPOSITION 4.1. *Let* $K_i \in \mathcal{K}(X)$, $i = 1, \ldots, m$ *and* $\bigcap_{i=1}^m U(K_i) \neq \emptyset$. *Let* $K = \bigcup_i K_i$. *Then* $K \in \mathcal{K}(X)$ *and*

$$\bigcap_{i=1}^m U(K_i) \subset U(K).$$

If $u \in \bigcap_{i=1}^m U(K_i)$ *then* $p_{u,K}(x) = \min_i p_{u,K_i}(x)$.

Proof. Let $u \in \bigcap_{i=1}^m U(K_i)$. Consider an element $x \in X$ and sets

$$\Lambda_x^i = \{\lambda \colon \lambda u - x \in K_i\}, \quad i = 1, \ldots, m, \quad \Lambda_x = \{\lambda \colon \lambda u - x \in K\}.$$

Note that $\Lambda_x^i = [p_{u,K_i}(x), +\infty)$. It is easy to check that

$$\Lambda_x = \bigcup_i \Lambda_x^i. \tag{4.1}$$

Indeed, if $\lambda \in \Lambda_x$, then $\lambda u - x \in K$ hence there exists i such that $\lambda u - x \in K_i$, hence $\lambda \in \bigcup_i \Lambda_x^i$. The same argument shows that the opposite inclusion holds. Consider the function $p = p_{u,K}$ defined by (3.1). It follows from (4.1) that

$$p(x) = p_{u,K}(x) = \min\{\lambda \colon \lambda \in \Lambda_x\} = \min\left\{\lambda \colon \lambda \in \bigcup_i \Lambda_x^i\right\} = \min_i p_{u,K_i}(x).$$

The function p is positively homogeneous and continuous as the minimum of positively homogeneous continuous functions. We also have

$$p(x + \lambda u) = \min_i p_{u,K_i}(x + \lambda u) = \min_i (\lambda + p_{u,K_i}(x)) = \lambda + p(x).$$

Thus p possesses the property (3.4). Note that

$$-K=\bigcup_i(-K_i)=\bigcup_i\{x\colon p_{u,K_i}(x)\leqslant 0\}=\left\{x\colon \min_i p_{u,K_i}(x)\leqslant 0\right\}.$$

Thus

$$K=\{-x\colon p(x)\leqslant 0\}.$$

It follows from the aforesaid and Theorem 3.1 that $K\in\mathcal{K}(X)$ and $u\in U(K)$. $\square$

COROLLARY 4.1. *Let* K_i, $i=1,\dots,m$ *be convex closed comes such that* $\bigcap_{i=1}^m \operatorname{int} K_i$ *is nonempty. Then* $\bigcup_{i=1}^m K_i\in\mathcal{K}(X)$.

Let K_i, $i=1,\dots,m$ be convex closed cones. The cone K_i generates the order relation $\geqslant_{K_i}$. The relation $\geqslant_K$, which is generated by the cone K, admits the following interpretation: $x\geqslant_K y$ if and only if there exists $i\in\{1,\dots,m\}$ such that $x\geqslant_{K_i} y$. Clearly $\geqslant_K$ is not a pre-order relation, if $\bigcup_i K_i$ is not convex.

We now give some examples of the functions $p_{u,K}$.

EXAMPLE 4.1. Let $C(T)$ be a space of continuous functions defined on a compact topological space T. Let K be the closed convex cone of nonnegative functions:

$$K=\{x\in C(Q)\colon x(t)\geqslant 0 \text{ for all } t\in T\}=\left\{x\in C(Q)\colon \min_{t\in T} x(t)\geqslant 0\right\}.$$

Then $\operatorname{int}K=\{x\in K\colon x(t)>0\}$ for all $t\in T$. Let $u\in\operatorname{int}K$. Then

$$\begin{aligned} p_{u,K}(x)&=\min\{\lambda\colon \lambda u(t)\geqslant x(t)\ \forall t\in T\}\\ &=\min\left\{\lambda\colon \lambda\geqslant \max_{t\in T}\frac{x(t)}{u(t)}\right\}=\max_{t\in T}\frac{x(t)}{u(t)}. \end{aligned}$$

In particular, if $u(t)=1$ for all $t\in T$, then $p_{u,K}(x)=\max_{t\in T} x(t)$.

EXAMPLE 4.2. Let $X=\mathbb{R}^n$, K coincides with the cone $\mathbb{R}^n_+$ of vectors with nonnegative coordinates. Then for each $u=(u_1,\dots,u_n)$ with $u_i>0$, $i=1,\dots,n$, we have:

$$p_{u,K}(x)=\max_{i=1,\dots,n}\frac{x_i}{u_i}.$$

It follows from the previous example, since $\mathbb{R}^n=C(T)$ with $T=\{1,\dots,n\}$.

EXAMPLE 4.3. Let $X = C(T)$ and let $K \subset C(T)$ be the set of all functions x such that $\max_{t\in T} x(t) \geqslant 0$ (in other words, the complement of K consists of all negative functions). The set K is not convex. Let $u(t) > 0$ for all t. Then

$$\begin{aligned} p_{u,K}(x) &= \min\{\lambda\colon \text{ there exists } t \in T \text{ such that } \lambda u(t) - x(t) \geqslant 0\} \\ &= \min\left\{\lambda\colon \text{ there exists } t \in T \text{ such that } \lambda \geqslant \frac{x(t)}{u(t)}\right\} \\ &= \min\left\{\lambda\colon \lambda \geqslant \min_{t\in T} \frac{x(t)}{u(t)}\right\} = \min_{t\in T} \frac{x(t)}{u(t)}. \end{aligned}$$

We can obtain the same result using Proposition 3.5 and the fact that $K = CL$, where $L = \{x\colon \max_{i\in T} x(t) \leqslant 0\}$.

Let $T = \{1, \dots, n\}$. Then $C(T) = \mathbb{R}^n$, $K = \{x = (x_1, \dots, x_n)\colon \max_i x_i \geqslant 0\}$. If $u \in \operatorname{int} \mathbb{R}^n_+$ then

$$p_{u,K} = \min_{i=1,\dots,n} \frac{x_i}{u_i}.$$

EXAMPLE 4.4. Let X be a normed space and $p\colon X \to \mathbb{R}$ be a sublinear (that is, positively homogeneous and convex) function. Then the epigraph

$$\operatorname{epi} p = \{(x, \alpha) \in X \times \mathbb{R}\colon \alpha \geqslant p(x)\}$$

is a closed convex cone. It is easy to check that the point $u = (0, 1)$ belongs to the interior of K. Let us calculate $p_{u,K}$. We have

$$\begin{aligned} p_{u,K}(x, \alpha) &= \inf\{\lambda\colon \lambda(0,1) - (x, \alpha) \in K\} = \inf\{\lambda\colon (-x, \lambda - \alpha) \in K\} \\ &= \inf\{\lambda\colon \lambda - \alpha \geqslant p(-x) = \inf\{\lambda\colon \lambda \geqslant \alpha + p(-x)\} = \alpha + p(-x). \end{aligned}$$

Consider now m sublinear functions $p_1, \dots, p_m$. Let $K = \bigcup_{i=1}^m \operatorname{epi} p_i$. Since $u = (0, 1) \in \bigcap_{i=1}^m \operatorname{int} \operatorname{epi} p_i$, we can apply Proposition 4.1. Due to this Proposition

$$p_{u,K}(x, \alpha) = \min_{i=1,\dots,m} (\alpha + p_i(-x)) = \alpha + \min_{i=1,\dots,m} p_i(x).$$

EXAMPLE 4.5. Consider the space $\mathbb{R}^n$ with $n \geqslant 2$. We now define two families of convex closed cones: $(K^\varepsilon)_{0\leqslant\varepsilon\leqslant\frac{1}{n}}$ and $(L^\varepsilon)_{0\leqslant\varepsilon\leqslant\frac{1}{n}}$. Let $0 \leqslant \varepsilon < 1/n$ and let $a_\varepsilon = 1 - (n-1)\varepsilon$ Consider the $n \times n$ matrix

$$A_\varepsilon = \begin{pmatrix} a_\varepsilon & \varepsilon & \dots & \varepsilon \\ \varepsilon & a_\varepsilon & \dots & \varepsilon \\ \cdot & \cdot & \dots & \cdot \\ \varepsilon & \varepsilon & \dots & a_\varepsilon \end{pmatrix}$$

An easy calculation shows that there exists the matrix $A_\varepsilon^{-1} := B_\varepsilon$ and

$$B_\varepsilon = \begin{pmatrix} b_\varepsilon & \delta_\varepsilon & \dots & \delta_\varepsilon \\ \delta_\varepsilon & b_\varepsilon & \dots & \delta_\varepsilon \\ \cdot & \cdot & \dots & \cdot \\ \delta_\varepsilon & \delta_\varepsilon & \dots & b_\varepsilon \end{pmatrix},$$

where

$$b_\varepsilon = \frac{1-\varepsilon}{1-n\varepsilon}, \quad \delta_\varepsilon = -\frac{\varepsilon}{1-n\varepsilon}$$

Consider close convex cones $L^\varepsilon = A_\varepsilon(\mathbb{R}^n_+), K^\varepsilon = B_\varepsilon(\mathbb{R}^n_+)$. If $\varepsilon = 0$ then $L^\varepsilon = K^\varepsilon = \mathbb{R}^n_+$.

Let $\mathbf{1} = (1, \dots, 1)$, then $\sum_{i=1}^n x_i$ coincides with the inner product $[\mathbf{1}, x]$ of vectors $\mathbf{1}$ and x. An easy calculation shows that

$$\begin{aligned} L^\varepsilon &= \left\{ y = (1-n\varepsilon)x + \varepsilon[x, \mathbf{1}]\mathbf{1} \colon x \in \mathbb{R}^n_+ \right\} \\ \operatorname{int} L^\varepsilon &= \left\{ y = (1-n\varepsilon)x + \varepsilon[x, \mathbf{1}]\mathbf{1} \colon x \in \operatorname{int}\mathbb{R}^n_+ \right\}. \\ K^\varepsilon &= \left\{ z = \frac{1}{1-n\varepsilon}(x - \varepsilon[1, x]\mathbf{1} \colon x \in \mathbb{R}^n_+ \right\} \\ \operatorname{int} K^\varepsilon &= \left\{ z = \frac{1}{1-n\varepsilon}(x - \varepsilon[1, x]\mathbf{1} \colon x \in \operatorname{int}\mathbb{R}^n_+ \right\}. \end{aligned}$$

Using this formulae, we can check the following:

(1) $L^\varepsilon \subset \operatorname{int}\mathbb{R}^n_+ \cup \{0\}$ for each $\varepsilon \in (0, 1/n)$.
(2) $0 < \varepsilon_1 < \varepsilon_2 < 1/n \Longrightarrow L^{\varepsilon_1} \subset L^{\varepsilon_2}$;
(3) $\bigcup_{\varepsilon > 0} L^\varepsilon = \operatorname{int}\mathbb{R}^n_+$.
(4) $\operatorname{int} K^\varepsilon \supset \mathbb{R}^n_+ \setminus \{0\}$ for each $\varepsilon > 0$;
(5) $0 < \varepsilon_2 < \varepsilon_1 < 1/n \Longrightarrow K^{\varepsilon_1} \subset K^{\varepsilon_2}$;
(6) $\bigcap_{\varepsilon > 0} K^\varepsilon = \mathbb{R}^n_+$.

It follows from items (4)–(6) that for each convex closed cone K with $\operatorname{int} K \cup \{0\} \supset \mathbb{R}^n_+$ there exists $\varepsilon > 0$ such $\operatorname{int} K \cup \{0\} \supset K^\varepsilon$.

We now calculate functions p_{u,L^ε} for $u \in \operatorname{int} L^\varepsilon$ and p_{u,K^ε} for $u \in \operatorname{int} K^\varepsilon$.

Denote the coordinate-wise order relation by $\geqslant$. It follows from the equality $L^\varepsilon = A_\varepsilon(\mathbb{R}^n_+)$ that $x \in L^\varepsilon$ is equivalent to $B_\varepsilon x \geqslant 0$. Since

$$(B_\varepsilon x)_i = \frac{1}{1-n\varepsilon}(x_i - \varepsilon[\mathbf{1}, x])$$

we have for $u\in\operatorname{int}L^{\varepsilon}$ and $x\in\mathbb{R}^n$:

$$\begin{aligned} p_{u,L^{\varepsilon}}(x) &= \inf\{\lambda\colon \lambda u - x\in L^{\varepsilon}\} = \inf\{\lambda\colon B_{\varepsilon}(\lambda u - x)\geqslant 0\} && (4.2)\\ &= \inf\{\lambda\colon \lambda B_{\varepsilon}u\geqslant B_{\varepsilon}x\} = \inf\{\lambda\colon \lambda(B_{\varepsilon}u)_i\geqslant (B_{\varepsilon}x)_i,\ i=1,\dots,n\}\\ &= \max_{i=1,\dots,n}\frac{(B_{\varepsilon}x)_i}{(B_{\varepsilon}u)_i} = \max_{i=1,\dots,n}\frac{x_i-\varepsilon[\mathbf{1},x]}{u_i-\varepsilon[\mathbf{1},u]}. && (4.3)\end{aligned}$$

In the same manner the equality $K^{\varepsilon}=B_{\varepsilon}(\mathbb{R}^n_+)$ implies the equivalence of $x\in K^{\varepsilon}\Longleftrightarrow A_{\varepsilon}\geqslant 0$. Since

$$(A_{\varepsilon}x)_i=(1-n\varepsilon)x_i+\varepsilon[1,x]$$

we have for $u\in\operatorname{int}K^{\varepsilon}$ and $x\in\mathbb{R}^n$:

$$\begin{aligned} p_{u,K^{\varepsilon}}(x) &= \inf\{\lambda\colon \lambda u - x\in K^{\varepsilon}\} = \inf\{\lambda\colon A_{\varepsilon}(\lambda u - x)\geqslant 0\} && (4.4)\\ &= \inf\{\lambda\colon \lambda A_{\varepsilon}u\geqslant A_{\varepsilon}x\} = \inf\{\lambda\colon \lambda(A_{\varepsilon}u)_i\geqslant (A_{\varepsilon}x)_i,\ i=1,\dots,n\}\\ &= \max_{i=1,\dots,n}\frac{(A_{\varepsilon}x)_i}{(A_{\varepsilon}u)_i} = \max_{i=1,\dots,n}\frac{(1-n\varepsilon)x_i+\varepsilon[\mathbf{1},x]}{(1-n\varepsilon)u_i+\varepsilon[\mathbf{1},u]}. && (4.5)\end{aligned}$$

5. Characterization of Minimal Points

Let $K\in\mathcal{K}(X)$. We need the following definitions

DEFINITION 5.1. Let S be a closed subset X. Then

(1) A point $\bar{s}\in S$ is called a weakly minimal point w.r.t K if $(\bar{s}-\operatorname{int}K)\cap S=\emptyset$.
(2) A point $\bar{s}\in S$ is called minimal w.r.t K if $(\bar{s}-K)\cap S=\{\bar{s}\}$.
(3) A point $\bar{s}$ is called proper minimal if there exists a closed conic set $L\in\mathcal{K}(X)$ such that $\operatorname{int}L\supset K\setminus\{0\}$, $U(L)\supset U(K)$ and $\bar{s}$ is a minimal point of S w.r.t. L.

Definition of weakly minimal (minimal, respectively) points coincide with classical definition of weakly Pareto-optimal (Pareto-optimal, respectively) points. Properly minimal points also well-known in literature. They were introduced by M. Henig in [2].

If S is an arbitrary subset of X then by definition, the sets of weakly minimal points of S, minimal points of S and properly minimal points of S coincide with the corresponding sets for the closure $\operatorname{cl}S$ of the set S; so in the sequel we shall consider only closed sets S.

We use the following notation: $\mathrm{W}-\mathrm{Min}_K S$ is the set of weakly minimal points of S, $\mathrm{Min}_K S$ is the set of minimal points of S and $P-\mathrm{Min}_K S$ is the set of properly minimal points of S.

PROPOSITION 5.1. *Let $K\in\mathcal{K}(X)$, S be a closed subset of X and $\bar{s}\in S$. Then*

(1) $\bar{s}\in W-\mathrm{Min}_K S$ *if and only if there exists $z\in X$ such that $u:=z+\bar{s}\in U(K)$ and $\bar{s}$ is a solution of the problem*

$$\min p_{u,K}(z+s) \quad \textit{subject to} \quad s\in S. \tag{5.1}$$

(2) $\bar{s}\in\mathrm{Min}_K S$ *if and only if there exists $z\in X$ such that $u:=z+\bar{s}\in U(K)$ and the problem* (5.1) *has the unique solution $z+\bar{s}$.*

Proof. We prove only item (2) of Proposition. The proof of the item (1) is similar. Let $\bar{s}$ be a minimal point of the set S. Let $z\in -\bar{s}+U(K)$. Then $u:=z+\bar{s}\in U(K)$. Consider the functional $p_{u,K}$. It follows from Proposition 3.1 (see 3.3) that $u-K=\{x\colon p_{u,K}(x)\leqslant 1\}$. Let $S_z=z+S$. Then u is a minimal element of S_z, that is $(u-K)\cap S_z=\{u\}$. It follows from aforesaid that $p_{u,K}(s_z)>1=p_{u,K}(u)$ for all $s_z\in S_z\setminus\{z+\bar{s}\}$. Hence

$$p_{u,K}(z+s)>p_{u,K}(z+\bar{s}), \quad s\in S\setminus\{\bar{s}\}.$$

Assume now that a vector $z\in X$ enjoy the following properties: $u=z+\bar{s}\in U(K)$ and $p_{u,K}(z+s)>p_{u,K}(z+\bar{s})=1$ for all $s\in S$, $s\neq\bar{s}$. It follows from (3.3) that $u-K=\{x\colon p_{u,K}(x)\leqslant 1\}$, hence $z+\bar{s}$ is a minimal element of $z+S$. Clearly $\bar{s}$ is a minimal element of S. Thus the result follows. □

PROPOSITION 5.2. *Let $K\in\mathcal{K}(X)$, S be a closed subset of X and $\bar{s}\in S$. Then the following assertions are equivalent.*

(i) $\bar{s}\in W-\mathrm{Min}_K S$.
(ii) *for each $u\in U(K)$ it holds that*

$$\min_{s\in S} p_{u,K}(\bar{s}-s)=0. \tag{5.2}$$

(iii) *there exists $u\in K(U)$ such that* (5.2) *holds.*

Proof. (i)$\Longrightarrow$(ii). By definition, $\bar{s}$ is a weakly minimal point if and only if for all $s\in S$ we have $\bar{s}-s\in X\setminus\mathrm{int}K$. The latter can be expressed in the form $\bar{s}-s\in CK$. Let $u\in U(K)$. Then due to Corollary 3.2 $-u\in U(CK)$. We have also $p_{-u,CK}(\bar{s}-s)\leqslant 0$. Applying Proposition we conclude that $p_{u,K}(\bar{s}-s)\geqslant 0$. Thus $\bar{s}-s$ is a solution of the problem (5.2).

(ii)$\Longrightarrow$(iii) is trivial.

(iii)$\Longrightarrow$(i). Let $\in U(K)$ be a vector such that $p_{u,K}(\bar{s}-s)\geqslant 0$ for all $s\in S$. Then $p_{-u,CK}(\bar{s}-s)\leqslant 0$ for all $s\in S$, hence $\bar{s}-S\subset CK$. This means that $(\bar{s}-S)\cap\mathrm{int}K$ is empty. □

PROPOSITION 5.3. *Let $K\in\mathcal{K}(X)$, S be a closed subset of X and $\bar{s}\in S$. Then the following assertions are equivalent*:

(i) $\bar{s}\in\mathrm{Min}_K S$.

(ii) *for each* $u \in U(K)$ *we have*

$$p_{u,K}(\bar{s}-s) > 0 \quad s \in S,\ s \neq \bar{s}. \tag{5.3}$$

(iii) *there exists* $u \in U(K)$ *such that* (5.3) *holds.*

Proof. The proof is similar to that of Proposition 5.2 and we omit it. □

We now turn to properly minimal points of S. Denote by $\mathcal{L}(K)$ the collection of all conic sets $L \in \mathcal{K}(X)$ such that $\operatorname{int} L \supset K \setminus \{0\}$ and $U(L) \supset U(K)$.

PROPOSITION 5.4. *Let* $K \in \mathcal{K}(X)$, *S be a closed subset of X and* $\bar{s} \in S$. *Then* $\bar{s} \in P-\mathrm{Min}_K S$ *if and only if there exist a conic set* $L \in \mathcal{L}(K)$ *and an element* $z \in X$, *which enjoy properties*: $u := z + \bar{s} \in U(K)$ *and $\bar{s}$ is a solution of problem*

$$\min p_{u,L}(s) \quad \textit{subject to } s \in S. \tag{5.4}$$

Proof. Let $\bar{s}$ be a proper minimal point of the set S. Then there exists $L \in \mathcal{L}(K)$ such that $\bar{s}$ is minimal element of S w.r.t L. Since $U(K) \subset U(L)$ we have $p_{u,L} \in \mathcal{P}_u$ if $u \in U(K)$. Taking $z \in -\bar{s} + U(K)$ and repeating the proof of Proposition 5.1 we conclude that $\bar{s}$ is a solution of (5.4).

Consider now $\bar{s} \in S$, $L \in \mathcal{L}(K)$ and $z \in X$ such that $u = z + \bar{s} \in U(K)$ and

$$1 = p_{u,L}(u) = \min_{s \in z+S} p_{u,L}(s). \tag{5.5}$$

It follows from the definition of $\mathcal{L}(K)$ that $u \in U(L)$ so $p_{u,L} \in \mathcal{P}_u$.

We have $\{x\colon p_{u,L}(x) \leqslant 1\} = u - L$. Since $\operatorname{int} L \supset K \setminus \{0\}$ we have $p_{u,K}(x) < 1$ for all $x \in K$. Let $p = (1/2)(p_{u,K} + p_{u,L})$. The set $\mathcal{P}_u$ is convex so $p \in \mathcal{P}_u$. We have $p(x + \mu u) = p(x) + \mu$ for all $x \in X$ and $\mu \in \mathbb{R}$. Let $L' = -\{x \in X\colon p(x) \leqslant 0\}$. Due to Theorem 3.1 we conclude that $L' \in \mathcal{L}(K)$, $u \in U(L')$ and $p = p_{u,L'}$. Let $x \in -K$, $x \neq 0$. Then $p_{u,K}(x) \leqslant 0$ and $p_{u,L}(x) < 0$, hence $p(x) = p_{u,L'}(x) < 0$. This means that $\operatorname{int} L' \supset K \setminus \{0\}$. It follows from Proposition 3.2 that $p_{u,L}(x) < p_{u,K}(x)$ for all $x \neq cu$ with $c \in \mathbb{R}$. Since $p(x) = (1/2)(p_{u,L}(x) + p_{u,K}(x) \leqslant 0$ for $x \in X$ it follows that $p_{u,L}(x) < 0$ for all $x \in L'$, $x \neq cu$. Let $cu \in -L'$, $cu \neq 0$. Then $c < 0$ so $p_{u,L'}(cu) = c < 0$. Hence $p_{u,L'}(x) < 0$ for all $x \in -L'$, $x \neq 0$. Due to (5.5) we conclude that $(z + \bar{s} - L') \cap (z + S) = \{z + \bar{s}\}$. This means that $z + \bar{s}$ is a minimal element of $z + S$ w.r.t L'. It follows from this result that $\bar{s}$ is a proper minimal element of S w.r.t. K. □

PROPOSITION 5.5. *Let* $K \in \mathcal{K}(X)$, *S be a closed subset of X and* $\bar{s} \in S$. *Then the following assertions are equivalent*:

(i) $\bar{s} \in \mathrm{Min}_K S$.

(ii) *there exists a conic set* $L \in \mathcal{L}(K)$ *such that for each* $u \in U(K)$ *we have*

$$\min_{s\in S} p_{u,L}(\bar{s}-s) > 0 \quad s \in S,\ s \neq \bar{s}. \tag{5.6}$$

(iii) *there exist a conic set* $L \in \mathcal{L}(K)$ *and an element* $u \in U(K)$ *such that* (5.6) *holds.*

The proof is similar to that of Proposition 5.2 and we omit it. □

Let $K \in \mathcal{K}(X)$. Denote by $\mathcal{L}_0(K)$ a collection of closed sets $L \in \mathcal{L}(K)$ such that for each $L \in \mathcal{L}(K)$ there exists $L' \in \mathcal{L}_0(K)$ with the property $L' \subset L$. Clearly a point $\bar{s}$ of a set $S \subset X$ is a proper minimal point of this set w.r.t. $K \in \mathcal{K}(X)$ if and only if there exists $L \in \mathcal{L}_0(K)$ such that $\bar{s}$ is a minimal point of S w.r.t. L. We can use only cones from the family $\mathcal{L}_0(K)$ in Proposition 5.4 and Proposition 5.2.

EXAMPLE 5.1. Let $X = \mathbb{R}^n$, $K = \mathbb{R}^n_+$. Consider a set $S \subset X$, which is bounded from below. This means that there exists $x \in \mathbb{R}^n$ such that $s \geqslant x$ for all $s \in S$. By adding a big enough vector x we can shift S to the interior $\operatorname{int}\mathbb{R}^n_+$ of the cone $\mathbb{R}^n_+$. We assume that the set S itself is located in $\operatorname{int}\mathbb{R}^n_+$. In such a case the zero can be chosen as an element z in Proposition 5.4. Consider the family $\mathcal{L}_0(K) = (K^\varepsilon)_{0<\varepsilon<\frac{1}{n}}$, where K^ε is the cone from Example 4.5. It follows from Proposition 5.4 that a point $u \in S$ is a proper minimal point of S if and only if there exist $\varepsilon > 0$ such that u is a solution of the problem

$$\min p_{u,K^\varepsilon}(s) \quad \text{subject to } s \in S. \tag{5.7}$$

It follows from (4.5) that (5.7) can be presented in the form

$$\min \max_{i=1,\dots,n} \frac{(1-n\varepsilon)s_i + \varepsilon[\mathbf{1}, s]}{(1-n\varepsilon)u_i + \varepsilon[\mathbf{1}, u]} \quad \text{subject to } s \in S.$$

6. Minimal Points w.r.t. Convex Cones

Characteristics of different types of minimal points become simpler in the case of convex cones K.

Let $K \subset X$ be a convex closed cone. Then $K \in \mathcal{K}(X)$ if $\operatorname{int}K$ is non empty. If $K \in \mathcal{K}(X)$ then $U(K) = \operatorname{int}K$. It is easy to see that the function $p_{u,K}$ is sublinear for each $u \in \operatorname{int}K$. Let $p \in \mathcal{P}_u$ and $-K = \{x\colon p(x) \leqslant 0\}$. Then K is convex if and only if p is sublinear. The functions $p_{u,K}$ with convex K have been studied by A. Rubinov [7] and D.T. Luc [5].

Denote by $\mathcal{P}$ the set of all continuous sublinear functions defined on X and such that there exists a point $x \in X$ for which $p(x) < 0$.

PROPOSITION 6.1. *Let* $p \in \mathcal{P}$ *be a function such that* $p(x) \leqslant 0$ *for all* $x \in -K$. *Then*

(1) *p is increasing with respect to* K: $x \geqslant_K y \Longrightarrow p(x) \geqslant p(y)$.

(2) *$p(x) < 0$ for $x \in \operatorname{int} K$.*
(3) *if $p(x) < 0$ for all $x \in -K \setminus \{0\}$, then p is strictly increasing: $x >_K y \Longrightarrow p(x) > p(y)$.*

Proof. Let $x \geqslant_K y$, that is $y = x + k$ where $k \in -K$. Then $p(y) \leqslant p(x) + p(k) \leqslant p(x)$. The same argument shows that the second and third parts of proposition hold. □

Remark 6.1. If K is not convex then the inequality $p_{u,K}(x) \geqslant p_{u,K}(y)$ does not necessarily hold if $x \geqslant_K y$.

Let S be a closed subset of X and $\bar{s} \in S$. In Section 5 we gave necessary and sufficient conditions for either weak minimality or minimality or proper minimality of $\bar{s}$ in terms of functions which depends on $\bar{s}$. If K is convex then sufficient conditions of weak minimality, minimality and proper minimality can be easily done in terms of functions $p \in \mathcal{P}$ that does not depend on the point $\bar{s}$.

PROPOSITION 6.2. *Let $K \subset X$ be a convex closed cone in a Banach space X, S be a closed subset X and $s \in S$. Let $p \in \mathcal{P}$ be a function such that $p(x) \leqslant 0$ for all $x \in -K$. Then*

(1) *if $p(\bar{s}) = \min\{p(s) \colon s \in S\}$ then $\bar{s} \in W - \mathrm{Min}_K S$.*
(2) *if $p(\bar{s}) < p(s)$ for all $s \in S$, $s \neq \bar{s}$ then $\bar{s} \in \mathrm{Min}_K S$.*
(3) *if $p(x) < 0$ for all $x \in K \setminus \{0\}$ and $p(\bar{s}) = \min\{p(s) \colon s \in S\}$ then $\bar{s} \in P - \mathrm{Min}_K S$.*

Proof. (1) It follows from Proposition 6.1 that $p(u) < 0$ for $u \in \operatorname{int} K$. Let $x \in \bar{s} - \operatorname{int} K$, then $x = \bar{s} + u$ with $u \in -\operatorname{int} K$, hence $p(x) \leqslant p(\bar{s}) + p(u) < p(\bar{s})$. This means that $x \notin S$. Thus $\bar{s}$ is a weakly minimal point.
(2) Let $x = \bar{s} - u$ with $u \in -K$. Then $p(x) \leqslant p(\bar{s}) + p(-u) \leqslant p(\bar{s})$. Hence $S \cap (\bar{s} - K) = \{\bar{s}\}$ and $\bar{s} \in \mathrm{Min}_K S$.
(3) Let $L = \{x \colon p(x) \leqslant 0\}$. Since $p(x) < 0$ for all $x \in K$, $x \neq 0$ it follows that $\operatorname{int} L \subset K \setminus \{0\}$. Let L' be a convex closed cone such that

$$\operatorname{int} L \supset L' \setminus \{0\} \supset \operatorname{int} L' \supset K \setminus \{0\}.$$

Then $p(x) < 0$ for all $x \in L'$, $x \neq 0$. Let $x \in \bar{s} - L'$, $x \neq \bar{s}$ that is, $x = \bar{s} + u$ with $u \in -L'$, $u \neq 0$. Then $p(x) < p(\bar{s})$. This means that $\bar{s} \in \mathrm{Min}_{L'} S$, hence $\bar{s}$ is a properly minimal element of S. □

Remark 6.2. If the space X in Proposition 6.2 is separable then there exists a sublinear continuous function p such that $p(x) < 0$ for all $x \in -K$, $x \neq 0$. Indeed, it is well-known that in this situation there exists a linear continuous function l such that $l(x) > 0$ for all $x \in K$, $x \neq 0$. Since $p_{u,K}$ with $u \in \operatorname{int} K$ enjoys the property

$p(x) \leqslant 0$ for all $x \in -K$, it follows that $p(x) := p_{u,K}(x) - l(x) < 0$ for all $x \in K$, $x \neq 0$.

Remark 6.3. An analogue of Proposition 6.2 does not hold for nonconvex cones $K \in \mathcal{K}(X)$.

7. Vector Optimization

In this section we consider some problems of vector optimization. For the sake of definiteness we consider only problems related to the weak minimality. Corresponding results can be easily extended to the case of minimal and properly minimal points. Let X be a metric space and Y, Z be Banach spaces. Let $K \in \mathcal{K}(Y)$ and $L \in \mathcal{K}(Z)$. Consider mappings F: $X \to Y$ and G: $X \to Z$ and the following problem $P(F,G)$: to find a weakly optimal point of the set $S := F(X_0)$ where

$$X_0 = \{x\colon G(x) \leqslant_L 0\} \tag{7.1}$$

We shall present this problem in the following form:

$$\mathrm{W-Min}_K F(x) \quad \text{subject to } G(x) \leqslant_L 0. \tag{7.2}$$

We consider two ways of scalarization of $P(F,G)$.

(1) Let $\bar{x} \in X_0$ be a solution of (7.2), that is $\bar{s} := F(\bar{x})$ is a weak minimal point of $S = F(X_0)$. Assume that $\bar{s} \in U(K)$, otherwise we can replace S with $z + S$ where $z \in -\bar{s} + U(K)$. Consider the function $f_{\bar{s}}(x) = p_{\bar{s},K}(F(x))$ and the problem of scalar minimization

$$\min f_{\bar{s}}(x) \quad \text{subject to } x \in X_0. \tag{7.3}$$

It follows from Proposition 5.1 that $\bar{x}$ is a solution of this problem. Indeed, due to this proposition we have

$$\begin{aligned} f_{\bar{s}}(\bar{x}) &= p_{\bar{s},K}(F(\bar{x})) = p_{\bar{s},K}(\bar{s}) = \min_{s \in S} p_{\bar{s},K}(s) \\ &= \min_{x \in X_0} p_{\bar{s},K}(F(x)) = \min_{x \in X_0} f_{\bar{s}}(x). \end{aligned} \tag{7.4}$$

We can present Problem (7.3) as the usual mathematical programming problem if we express the set X_0 of feasible elements as the set of solutions of a system of inequality constraints. Since the solution set of an arbitrary system of inequalities $h_t(x) \leqslant 0 (t \in T)$ is also the solution set of a single inequality $h(x) \leqslant 0$ (for example, we can consider $h(x) = \sup_{t \in T} h_t(x)$) we express X_0 as the solution set of a single inequality. We can use for this purpose the function $p_{v,L}$ where $v \in U(L)$. Let $g_v(x) = p_{v,L}(G(x))$. Then the inequality $G(x) \leqslant_L 0$ is equivalent to

$g_v(x) \leqslant 0$ and problem (7.3) is equivalent to the following problem $P(f_{\bar{s}}, g_v)$ of mathematical programming with a single constraint:

$$\min f_{\bar{s}}(x) \quad \text{subject to } x \in X, g_v(x) \leqslant 0 \tag{7.5}$$

THEOREM 7.1. *Consider Problem $P(F, G)$ and an element $\bar{x} \in X$ such that $\bar{s} = F(\bar{x}) \in U(K)$. Then the following assertions are equivalent*:

(1) *$\bar{x}$ is a solution of $P(F, G)$;*
(2) *$\bar{x}$ is a solution of $P(f_{\bar{s}}, g_v)$, where $\bar{s} = F(\bar{x})$, $f_{\bar{s}}(x) = p_{\bar{s},K}(F(x))$ and $g_v(x) = p_{v,L}(x)$ with an arbitrary $v \in U(L)$.*

Proof. We need to prove only $(2) \Longrightarrow (1)$. Let $\bar{x}$ be a solution of $P(f_{\bar{s}}, g_v)$. Then due to the definition of $f_{\bar{s}}$ and g_v we have $p_{\bar{s},K}(\bar{s}) = \min\{p_{\bar{s},K}(s)\colon s \in F(X_0)\}$. Applying Proposition 5.1 we deduce that $\bar{s} \in \text{W-Min}\, F(X_0)$. □

2. Let $\bar{x}$ be a solution of $P(F, G)$. Let u be an arbitrary element of $U(K)$. Consider a function $\tilde{f}_{\bar{x},u}(x) = p_{u,K}(F(x) - F(\bar{x}))$ and a function $g(x) = p_{v,L} G(x)$ with an arbitrary $v \in U(L)$. The following result follows directly from Proposition 5.2.

THEOREM 7.2. *Consider Problem $P(F, G)$ and an element $\bar{x} \in X$. Then the following assertions are equivalent*:

(1) *$\bar{x}$ is a solution of $P(F, G)$;*
(2) *$\bar{x}$ is a solution of $P(\tilde{f}_{\bar{x},u}, g_v)$, where u, v, $\tilde{f}_{\bar{x},u}$ and g_v as above.*

Sometimes it is convenient to have a scalar optimization problem $P(f, g)$ with the positive objective function f. We shall consider only optimization problems $P(F, G)$ such that for each $\bar{x} \in X_0$ the set $\{F(x) - F(\bar{x})\colon x \in X_0\}$ is bounded from below in the sense of K. This means that there exist $y \in Y$ such that $F(x) - F(\bar{x}) \geqslant_K y$ for all $x \in X_0$. Let $u \in U(K)$. Due to Proposition 2.1 there exists $\nu \in \mathbb{R}$ such that $F(x) - F(\bar{x}) \geqslant_K \nu u$ for all $x \in X_0$, hence $p_{u,K}(F(x) - F(\bar{x}) \geqslant \nu$. Since $\bar{x} \in X_0$ it follows that $\nu \leqslant 0$. Let

$$\hat{f}_{\bar{x},u(x)} = \max(p_{u,K}(F(x) - F(\bar{x})) - \nu, 0) + 1, \quad x \in X \tag{7.6}$$

Clearly $\inf_{x \in X} \hat{f}_{\bar{x},u}(x) \geqslant 1$. We also have for $x \in X_0$:

$$\hat{f}_{\bar{x},u} u(x) = p_{u,K}(F(x) - F(\bar{x})) - \nu + 1 = \tilde{f}_{\bar{x},u}(x) - \nu + 1.$$

Hence problems $P(\tilde{f}_{\bar{x},u}, g_v)$ and $P(\hat{f}_{\bar{x},u}, g_v)$ are equivalent in the sense that both of these problems have the same set of local and global minima. Applying Theorem 7.2 we conclude that the following result holds.

THEOREM 7.3. *Consider problem $P(F, G)$ and an element $\bar{x} \in X$. Assume that the set $\{F(x) - F(\bar{x})\colon x \in X_0\}$ is bounded from below and consider the function*

$\hat{f}_{\bar{x},u}$ defined by (7.6) *and the function g_v with $v \in U(L)$. Then the following assertions are equivalent*:

(1) *$\bar{x}$ is a solution of $P(F, G)$;*
(2) *$\bar{x}$ is a solution of $P(\hat{f}_{\bar{x},u}, g_v)$.*

Theorems 7.1, 7.2 and 7.3 reduce problem $P(F, G)$ to scalar optimization problems and we can use some known duality results for scalar optimization for examination of $P(F, G)$.

8. Scalar Duality

A dual theory based on nonlinear penalization has been recently developed for scalar optimization problems with a single constraint. (See [10], Chapter 4 and references therein.) We present here some results of this theory.

We need the so-called IPH (increasing positively homogeneous) functions h: $\mathbb{R}^2_+ \to \mathbb{R}$, where $\mathbb{R}^2_+ = \{z = (z_1, z_2) \in \mathbb{R}^2 \colon z_1 \geqslant 0, z_2 \geqslant 0\}$. A function h: $\mathbb{R}^2_+ \to \mathbb{R}$ is called IPH if h is increasing with respect to coordinate-wise order relation and positively homogeneous of degree one: $h(\lambda z) = \lambda h(z)$ for $x \in \mathbb{R}^2_+$ and $\lambda > 0$. Denote by $\mathcal{H}$ the class of IPH functions h such that

$$h(1,0) = 1, \quad \lim_{z_2 \to +\infty} h(1, z_2) = +\infty.$$

Consider a problem $P(f, g)$:

$$\min f(x) \quad \text{subject to } g(x), \leqslant 0 \tag{8.1}$$

where f, g: $X \to \mathbb{R}$ and X is a metric space. We assume that $\inf_{x \in X} f(X) > 0$. This assumption is not very restrictive (see the discussion in [10], Chapter 3). Let $h \in \mathcal{H}$. The function $L^+_h(x, d)$ defined by

$$L^+_h(x, d) = h(f(x),\ dg^+(x)), \quad x \in X,\ d \in \mathbb{R}_+ \tag{8.2}$$

is called the penalty-type function of $P(f, g)$ generated by h. Here $g^+(x) = \max(g(x), 0), \mathbb{R}_+ = \{x \in \mathbb{R} \colon x \geqslant 0\}$. If $h_1(z_1, z_2) = z_1 + z_2$ then $L^+_{h_1}$ coincides with the classical penalty function: $L^+_{h_1}(x, d) = f(x) + dg^+(x)$. Using the penalty-type function L^+_h with $h \in \mathcal{H}$ we can define the dual function $q_h(d)$ and the dual problem $D_h(f, g)$. By definition

$$q_h(d) = \inf_{x \in X} L^+_h(x, d) \equiv \inf_{x \in X} h(f(x),\ dg^+(x)).$$

Dual problem $D_h(f, g)$ has the form

$$\sup_{d \geqslant 0} q_h(d) \equiv \sup_{d \geqslant 0} \inf_{x \in X} h(f(x),\ dg^+(x)) \tag{8.3}$$

If $h \in \mathcal{H}$ then the *weak duality* property holds: $\sup_{d>0} q_h(d) \leqslant \inf_{x \in X_0} f(x)$ where $X_0 = \{x \in X \colon g(x) \leqslant 0\}$. The following questions are of interest:

(1) When the zero duality gap property holds, that is, $\sup_{d \geq 0} q_h(d) = \inf_{x \in X_0} f(x)$?
(2) When does the exact parameter $\bar{d}$ exist? (A number $\bar{d}$ is the exact parameter if $q_h(\bar{d}) = \inf_{x \in X_0} f(x)$).

It follows from Theorem 8.1 below that the zero duality gap property depends only on problem $P(f,g)$ and does not depend on the exogenous funcions h. However the existence of an exact parameter heavily depends on h. This is the main reason for examination of penalty-type functions with $h \in \mathcal{H}$. Here we consider only the zero duality gap property. The technique that is used here can also applied for examination of exactness: many results from [10] related to the exact parameters can be transformed for vector optimization.

For examination of the zero duality gap property we need the perturbation function β of $P(f,g)$. By definition,

$$\beta(t) = \inf\{f(x) \colon g(x) \leqslant t\}, \quad t \geqslant 0. \tag{8.4}$$

THEOREM 8.1 (see [10] and references therein). *Let $h \in \mathcal{H}$. Consider problem $P(f,g)$ such that $\inf_{x \in X} f(x) > 0$. Then the zero duality gap property $\sup_{d>0} q_h(d) = \inf_{x \in X_0} f(x)$ holds if and only if the perturbation function β of $P(f,g)$ is lower semicontinuous at the origin.*

We now come back to problem $P(F,G)$ defined by (7.2). For the sake of definiteness we use only the second approach to the scalarization of this problem. Let $u \in U(K)$ and $v \in U(L)$. Let $\bar{x} \in X$. Assume that the set $\{F(x) - F(\bar{x}) \colon x \in X_0\}$ is bounded from below by K and consider problem $P(\hat{f}_{\bar{x},u}, g_v)$ where $\hat{f}_{\bar{x},u}$ is defined by (7.6):

$$\hat{f}_{\bar{x},u}(x) = \max(p_{u,K}(F(x) - F(\bar{x})) - \nu, 0) + 1, \quad x \in X,$$

where ν is a number with the property $F(x) - F(\bar{x}) \geqslant_K \nu u$ for all $x \in X_0$.

Let $h \in \mathcal{H}$. Then the function

$$L_h^+(x,d) = h(\hat{f}_{\bar{x},u}(x), d g_v(x)), \quad x \in X, d \geqslant 0$$

is called the penalty-type function of the problem $P(F,G)$ corresponding to $\bar{x}, u$, v and h. The function

$$q_{\bar{x},u,v,h}(d) = \inf_{x \in X} h(\hat{f}_{\bar{x},u}(x), d g_v(x)), \quad d \geqslant 0$$

is called the dual function of $P(F,G)$ corresponding to the totality $(\bar{x}, u, v, h)$ and the problem $D_{\bar{x},u,v,h}(F,G)$:

$$\sup q_{\bar{x},u,v,h}(d) = \sup_{d \geqslant 0} q_{\bar{x},u,v,h}(d) \equiv \sup_{d \geqslant 0} \inf_{x \in X} h(\hat{f}_{\bar{x},u}(x), d g_v(x))$$

is called the dual problem of $P(F,G)$ corresponding to this totality.

THEOREM 8.2. *Consider problem* $P(F,G)$. *Let* $\bar{x}\in X$ *be a point such that the set* $\{F(x)-F(\bar{x})\colon x\in X_0\}$ *is bounded from below. Let* $u\in U(K), v\in U(L)$ *and* $h\in\mathcal{H}$. *Assume that the perturbation function* β *of* $P(\hat{f}_{\bar{x},u}g_v)$ *is lower semicontinuous at the origin. Then* $\bar{x}$ *is a solution of* $P(F,G)$ *if and only if*

$$\sup_{d\geqslant 0}\inf_{x\in X} h(\hat{f}_{\bar{x},u}(x),\mathrm{d}g^+(x))=-\nu+1 \tag{8.5}$$

Proof. It follows from Theorem 8.1 that

$$\sup_{d\geqslant 0}\inf_{x\in X} h(\hat{f}_{\bar{x},u}(x),\ dg^+(x))=\inf_{x\in X_0}\hat{f}_{\bar{x},u}(x)=\inf_{x\in X_0}\hat{f}_{\bar{x},u}(x) \tag{8.6}$$

Due to (7.6) we have $\hat{f}_{\bar{x},u}(x)=p_{u,K}(F(x)-F(\bar{x}))-\nu+1$ for $x\in X_0$. Let $\bar{x}$ be a solution of $P(F,G)$. Applying Theorem 7.3 we conclude that $\bar{x}$ is a solution of $P(\hat{f}_{\bar{x},u}g_v)$, hence

$$\inf_{x\in X_0}\hat{f}_{\bar{x},u}(x)=\inf_{x\in X_0}p_{u,K}(F(x)-F(\bar{x}))-\nu+1=\nu+1.$$

It follows from (8.6) that (8.5) holds. Assume now that (8.5) is valid. Combining (8.5) and (8.6) we deduce that

$$\inf_{x\in X_0}\hat{f}_{\bar{x},u}(x)=-\nu+1=\hat{f}_{\bar{x},u}(\bar{x}).$$

Hence $\bar{x}$ is a solution of $P(\hat{f}_{\bar{x},u}g_v)$. Applying again Theorem 7.3 we conclude that $\bar{x}$ is a solution of $P(F,G)$. □

In order to give sufficient conditions for lower semicontinuity of the perturbation function, we need to express this function in terms of the problem $P(F,G)$.

PROPOSITION 8.1. *Let* β *be the perturbation function of* $P(\hat{f}_{\bar{x},u}g_v)$. *Then*

$$\beta(t)=\inf_{x\colon G(x)\in tv-L}\max(p_{u,K}(F(x)-F(\bar{x})-\nu,0)+1).$$

Proof. We have for $t>0$:

$$\beta(t)=\inf_{x\colon g_v(x)\leqslant t}\hat{f}_{\bar{x},u}(x)=\inf_{x\colon p_{v,L}(G(x))\leqslant t}\hat{f}_{\bar{x},u}(x).$$

Due to Proposition 3.1 we have $p_{v,L}(G(x))\leqslant t\Longleftrightarrow G(x)\in tv-L$, hence

$$\beta(t)=\inf_{x\colon G(x)\in tv-L}\max(p_{u,K}(F(x)-F(\bar{x})-\nu,0)+1). \qquad \square$$

Consider the set-valued mapping D: $\mathbb{R}_+ \to 2^X$ defined by $D(t) = \{x\colon G(x) \in tv - L\}$. Note that $D(0) = X_0$. We assume that D is upper semicontinuous at the origin that is for each $\delta > 0$ there exists $\eta > 0$ such that $D(t) \subset \{x \in X\colon d(x, X_0) < \delta\}$ for all $f \in [0, \eta)$. Here $d(x, X_0) = \inf_{x_0 \in X_0} d(x, x_0)$ and $d(x, x_0)$ is the distance between x and x_0 in the metric space X.

PROPOSITION 8.2. *Let the mapping D is upper semicontinuous at the origin and the function $\hat{f}_{\bar{x},u}$ is uniformly continuous on the set X_0. Then the perturbation function β is lower semicontinuous at the origin.*

Proof. It follows from properties of $\hat{f}_{\bar{x},u}$ and D that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, x') < \delta$ implies $|f(x) - f(x')| < \varepsilon$ and there exist $\eta > 0$ such that $t \in (0, \eta)$ implies

$$D(t) = \{x\colon G(x) \in tv - L\} \subset \{x\colon d(x, X_0) < \delta\}$$

Let $t \in (0, \eta)$ and let $x' \in D(t)$ be a point such that $f(x') < \inf_{x \in D(t)} f(x) - \varepsilon$. Then for the point $x' \in D(t)$ there exists $x_0 \in X_0$ such that $d(x', x_0) < \delta$. It follows from the definition of δ that $|f(x') - f(x_0)| < \varepsilon$. Hence

$$\beta(t) - \inf_{x \in D(t)} f(x) \geqslant f(x') + \varepsilon \geqslant f(x_0) + 2\varepsilon \geqslant \inf_{x \in X_0} f(x) + 2\varepsilon = \beta(0) + 2\varepsilon. \qquad \square$$

Remark 8.1. The uniform continuity of $\hat{f}_{\bar{x},u}$ follows from uniform continuity of both F and $p_{u,K}$. Note that the function $p_{u,K}$ is uniformly continuous if K is a finite unit of convex closed cones K_i with the nonempty intersection of interiors of K_i.

Remark 8.2. If the set $D(t)$ is compact for some $t > 0$ then the lower semicontinuity of the perturbation function β follows from the lower semicontinuity of $\hat{f}_{\bar{x},u}$ and lower semicontinuity of g_v. We omit the simple proof of this assertion.

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